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Generic solutions for some perturbed optimization problem in non-reflexive Banach spaces ☆

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Abstract

Let Z be a closed, boundedly relatively weakly compact, nonempty subset of a Banach space X, and $J: Z \to R$ a lower semicontinuous function bounded from below. If X_0 is a convex subset in X and X_0 has approximatively Z-property (K), then the set of all points x in $X_0 \setminus Z$ for which there exists $z_0 \in Z$ such that $J(z_0) + \|x - z_0\| = \varphi(x)$ and every sequence $\{z_n\} \subset Z$ satisfying $\lim_{n\to\infty}[J(z_n) + \|x - z_n\|] = \varphi(x)$ for x contains a subsequence strongly convergent to an element of Z is a dense G_δ-subset of $X_0 \setminus Z$. Moreover, under the assumption that X_0 is approximatively Z-strictly convex, we show more, namely that the set of all points x in $X_0 \setminus Z$ for which there exists a unique point $z_0 \in Z$ such that $J(z_0) + \|x - z_0\| = \varphi(x)$ and every sequence $\{z_n\} \subset Z$ satisfying $\lim_{n\to\infty}[J(z_n) + \|x - z_n\| = \varphi(x)$ for x converges strongly to z_0 is a dense G_δ-subset of $X_0 \setminus Z$. Here $\varphi(x) = \inf\{J(z) + \|x - z\|;\ z \in Z\}$. These extend S. Cobzas's result [J. Math. Anal. Appl. 243 (2000) 344–356].

Keywords: Perturbed optimization problems; Lower semicontinuous function; Boundedly relatively weakly compact subset; Dense G_δ-subset

1. Introduction

This paper is concerned with the generic solutions for some perturbed optimization problems containing as a particular case the problem of nearest point. Let X be a real Ba-

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nach space with norm $\|\cdot\|$, Z a nonempty subset of X, and X^* be the dual of X. We use $B(x,r)$ to denote the closed ball in X with center x and radius r (>0). Throughout this paper the functional $J : Z \to R$ will be supposed bounded from below and lower semicontinuous (lsc).

For a functional $J : Z \to R$ and $x \in X$, consider the following problem [5]:

Problem *J*-inf: Find $z_0 \in Z$ such that $J(z_0) + \|x - z_0\| = \inf_{z \in Z}\{J(z) + \|x - z\|\}$.

Baranger [1] proved that if Z is a nonempty closed subset of a uniformly convex Banach space X, then the set of all $x \in X$ for which Problem J-inf has a solution is a dense G_δ-subset of X, containing Stechkin's results [14] as a particular case (the case $J = 0$). Recently, S. Cobzas [5] extends Baranger's results [1] mentioned above to reflexive Kadec Banach space. For other results on perturbed optimization problems, see [9,10,12,15] and the monograph [8].

The results on the generic solutions for perturbed optimization problems have been applied in [1–3,6] to optimal control problems governed by partial differential equations (PDE). If the control problem of minimization of the quantity $\|y(u) - z\|$ over a subset U (of admissible controls) of a Banach space X, where $y(u)$ is the unique solution of a PDE, has no solution, then one replaces it by a perturbed problem of minimization of the function $\|y(u) - z\| + \varepsilon\|u - v\|$ (or $\|y(u) - z\| - \varepsilon\|u - v\|$) to which the general generic existence results can be applied.

In the present paper, using different approach, which was developed by Lau [11] and Borwein and Fitzpatrick [4], we prove that if Z is a closed, boundedly relatively weakly compact, nonempty subset of a Banach space X, $J : Z \to R$ is a lower semicontinuous function bounded from below, X_0 is a convex subset in X and X_0 has approximatively Z-property (K), then the set

$$M(Z) \cap X_0 = \Big\{x \in X \setminus Z;\ \text{there exists } z_0 \in Z \text{ such that } J(z_0) + \|x - z_0\| = \varphi(x) \\ \text{and every sequence } \{z_n\} \subset Z \text{ satisfying} \\ \lim_{n\to\infty}\big[J(z_n) + \|x - z_n\|\big] = \varphi(x) \text{ contains a subsequence} \\ \text{strongly convergent to an element of } Z\Big\} \cap X_0$$

is a dense G_δ-subset of $X_0 \setminus Z$. Moreover, under the assumption that X_0 is approximatively Z-strictly convex, we show more, namely that the set

$$N(Z) \cap X_0 = \Big\{x \in X \setminus Z;\ \text{there exists a unique point } z_0 \in Z \text{ such that} \\ J(z_0) + \|x - z_0\| = \varphi(x) \text{ and every sequence } \{z_n\} \subset Z \\ \text{satisfying } \lim_{n\to\infty}\big[J(z_n) + \|x - z_n\|\big] = \varphi(x) \text{ converges} \\ \text{strongly to } z_0\Big\} \cap X_0$$

is a dense G_δ-subset of $X_0 \setminus Z$. Where $\varphi(x) = \inf_{z \in Z}\{J(z) + \|x - z\|\}$. These extend S. Cobzas's result [5].

2. Main result

Let Z be a closed nonempty subset of X. For $x \in X$ let

$$P(x) = \{z \in Z;\ J(z_0) + \|x - z_0\| = \varphi(x)\}.$$

Definition 2.1. X_0 is said to be approximatively strictly convex respect to Z (or approximatively Z-strictly convex) if for any $x \in X_0$ and $z_1, z_2 \in P(x)$, $\|x - z_1 + x - z_2\| = \|x - z_1\| + \|x - z_2\|$ implies $z_1 = z_2$. Where X_0 is a convex subset in X.

Definition 2.2. X_0 said to have approximatively property (K) with respect to Z (or approximatively Z-property (K)) if for any $x \in X_0$ and any sequence $\{z_n\} \in Z$ satisfying $\lim_{n\to\infty}[J(z_n) + \|x - z_n\|] = \varphi(x)$, with $\{z_n\}$ converges to $z_0 \in X$ weakly, that $\|z_n - z_0\| \to 0$ provided that $\lim_{n\to\infty} \|x - z_n\| = \|x - z_0\|$ and X_0 is a convex subset in X.

Clearly, for any two subsets X_0, Z in X, it follows that

(1) X is strictly convex $\Rightarrow$ X_0 is approximatively strictly convex with respect to Z.
(2) X has the Kadec property $\Rightarrow$ X_0 has the approximatively property (K) with respect to Z.

Locally uniformly convex Banach spaces and the space L_P $(1 < P < +\infty)$ have the Kadec property.

Definition 2.3 [13]. Let D be an open subset of X. A real-valued function f on D is said to be Fréchet differentiable at $x \in D$ if there exists $x^* \in X^*$ such that

$$\lim_{y\to x} \frac{f(y) - f(x) - \langle x^*, y - x\rangle}{\|y - x\|} = 0.$$

x^* is called the Fréchet differential at x which is denoted by $Df(x)$.

Lemma 2.1 [13]. *Let f be a locally Lipschitz continuous function on an open set D of a reflexive Banach space. Then f is Fréchet differentiable on a dense subset of D.*

As J is bounded from below on Z, say by $-a \in R$, it follows that $J(z) + a \geqslant 0$ $(z \in Z)$ and $\inf_{z\in Z}\{a + J(z) + \|x - z\|\} = a + \inf_{z\in Z}\{J(z) + \|x - z\|\}$, so that we can suppose $J(z) \geqslant 0$ $(z \in Z)$.

Lemma 2.2 [5, Lemma 2.2]. *The function φ satisfies the relation*

$$\left|\varphi(x) - \varphi'(x)\right| \leqslant \|x - x'\| \quad \textit{for all } x, x' \in X.$$

For $x \in X \setminus Z$ and $\delta > 0$, put

$$Z(x, \delta) = \{z \in Z;\ J(z) + \|x - z\|\varphi(x) + \delta\}.$$

Let

$$L_n(Z) = \Big\{ x \in X \setminus Z;\ \inf_{z \in Z(x,\delta)} \big\{ J(z) + \langle x^*, x - z \rangle \big\} > (1 - 2^{-n})\varphi(x) \textit{ for some } \delta > 0 \\ \textit{and some } x^* \in X^* \textit{ with } \|x^*\| = 1 \Big\}.$$

Also let $L(Z) = \bigcap_n L_n(Z)$ *and*

$$\Omega(Z) = \Big\{ x \in X \setminus Z;\ \textit{there exists } x^* \in X^* \textit{ with } \|x^*\| = 1 \textit{ such that} \\ \textit{for each } \varepsilon \in (0,1) \textit{ there is } \delta > 0 \textit{ such that} \\ \inf_{z \in Z(x,\delta)} \big\{ J(z) + \langle x^*, x - z \rangle \big\} > (1 - \varepsilon)\varphi(x) \Big\}.$$

Obviously, $\Omega(Z) \subset L(Z)$.

Lemma 2.3. $L(Z) \cap X_0$ *is a* G_δ*-subset of* $X_0 \setminus Z$.

Proof. To show that $L(Z) \cap X_0$ is a G_δ-subset of $X_0 \setminus Z$, we only need to prove that $L(Z) \cap X_0$ is open for each n. Let $x \in L_n(Z) \cap X_0$. Then there exist $x^* \in X^*$ with $\|x^*\| = 1$ and $\delta > 0$ such that

$$\tau = \inf_{z \in Z(x,\delta)} \big\{ J(z) + \langle x^*, x - z \rangle \big\} - (1 - 2^{-n})\varphi(x) > 0.$$

Let $\lambda > 0$ be such that $\lambda < \min(\delta/2, \tau/2)$ and fix y with $\|y - x\| < \lambda$. For $\delta^* = \delta - 2\lambda$, we have $Z(y, \delta^*) \subset Z(x, \delta)$. In fact, let $z \in Z(y, \delta^*)$; from Lemma 2.2, we have $J(z) + \|y - z\| < \varphi(y) + \delta^*$ and

$$\begin{aligned} J(z) + \|x - z\| &\leqslant J(z) + \|y - z\| + \|y - x\| < \varphi(y) + \lambda + \delta^* \\ &\leqslant \varphi(x) + \|y - x\| + \lambda + \delta^* < \varphi(x) + 2\lambda + \delta^* = \varphi(x) + \delta. \end{aligned}$$

Hence if $z \in Z(y, \delta^*)$, then $J(z) + \langle x^*, x - z \rangle \geqslant \tau + (1 - 2^{-1})\varphi(x)$ and

$$\begin{aligned} J(z) + \langle x^*, y - z \rangle &= J(z) + \langle x^*, x - z \rangle + \langle x^*, y - x \rangle \\ &\geqslant \tau + (1 - 2^{-n})\varphi(x) - \|x - y\| \\ &= \tau + (1 - 2^{-n})\varphi(y) - \|x - y\| - (1 - 2^{-n})\big[\varphi(y) - \varphi(x)\big] \\ &\geqslant \tau + (1 - 2^{-n})\varphi(y) - 2\|x - y\| \geqslant (1 - 2^{-n})\varphi(y) + \tau - 2\lambda. \end{aligned}$$

Thus $\inf_{z \in Z(y,\delta^*)}\{J(z) + \langle x^*, y - z \rangle\} > (1 - 2^{-n})\varphi(y)$ and $y \in L_n(Z) \cap X_0$ for all $y \in X$. With $\|y - x\| < \lambda$, which implies that $L_n(Z) \cap X_0$ is open in $X_0 \setminus Z$. □

The following lemma, due to Davis, Figiel, Johnson and Pelczynski [7], plays a key role in the proof of the density of $L(Z) \cap X_0$.

Lemma 2.4 [7]. *Let* K *be weakly compact subset of a Banach space* $Y = \overline{\operatorname{span} K}$. *Then there exists a reflexive Banach space* Q *and a one to one continuous linear mapping* $T : Q \to Y$ *such that* $T(B(0,1)) \supset K$.

Lemma 2.5. *Let* Z *be a closed, boundedly relatively weakly compact, nonempty subset of a Banach space* X, *then* $L(Z) \cap X_0$ *is dense in* $X_0 \setminus Z$.

Proof. From $\Omega(Z) \cap X_0 \subset L(Z) \cap X_0$, it suffices to prove that $\Omega(Z) \cap X_0$ is dense in $X_0 \setminus Z$. Let $x_0 \in X_0 \setminus Z$ and suppose $\varphi(x_0) > \varepsilon > 0$. Let

$$K = \text{week-cl}\big\{\big(B(0, N) \cap Z\big) \cup \{x_0\}\big\},$$

where $N = \|x_0\| + 3\varphi(x_0)$. Then K is weakly compact and if Y is the closed span of K, we can apply Lemma 2.4 to obtain a reflexive Banach space Q and a one-to-one continuous linear mapping $T : Q \to Y$ such that $T(B(0, 1)) \supset K$. Set

$$O_T = \big\{y = z + tx_0 \in Y;\ z \in \overline{\operatorname{span} K},\ 0 < t < 1\big\}.$$

Then O_T is open in Y so that $O = \{u \in Q,\ Tu \in O_T\}$ is open in Q.

Define $f_z : O \to [0, +\infty)$ by

$$f_z(u) = \varphi(Tu), \quad \forall u \in O.$$

Then $f_z(\cdot)$ is a Lipschitz function on Q and so by Lemma 2.1, $f_z(\cdot)$ is Fréchet differentiable on a dense subset of O. Thus there exists a differentiable point $v \in O$ of $f_z(\cdot)$ with $Df_z(v) = v^*$ such that $y = Tv \in \mathring{B}(x_0, \varepsilon)$. This means that

$$\lim_{h\to 0} \frac{\varphi(T(v + h)) - \varphi(Tv) - \langle v^*, h\rangle}{\|h\|} = 0$$

and hence

$$\lim_{h\to 0} \frac{\varphi(y + Th) - \varphi(y) - \langle v^*, h\rangle}{\|h\|} = 0.$$

Substituting tu ($\forall u \in Q$ and $t \to 0$) for h in the previous expression and using Lemma 2.2, we have $\langle v^*, u\rangle \leqslant \|Tu\|$. This shows $v^* = T^*y^*$ for some $y^* \in Y^*$. Furthermore, $\langle y^*, Tu\rangle \leqslant \|Tu\|$ for all $u \in Q$, so that $\|y^*\| < 1$ since T has dense range. By Hahn–Banach theorem, we may extend y^* to x^* with $\|x^*\| \leqslant 1$. Observe that

$$\lim_{t\to 0} \frac{\varphi(y + tTh) - \varphi(y) - t\langle v^*, Th\rangle}{t} = 0.$$

So that

$$\lim_{\substack{t\to 0\\ k\in K}} \left[\frac{\varphi[y + t(k - y)] - \varphi(y)}{t} \langle x^*, k - y\rangle\right] = 0.$$

(Since $T(B(0, 1) - v) \supset K - y$.) Now choose $\{z_n\} \subset Z$ satisfying

$$\lim_{n\to\infty} \big[J(z_n) + \|y - z_n\|\big] = \varphi(y) \tag{1}$$

for $y \in X_0 \setminus Z$. Then for each $t \in (0, 1)$,

$$\begin{aligned}\varphi\big[y + t(z_n - y)\big] - \varphi(y) &\leqslant J(z_n) + \|y + t(z_n - y) - z_n\| - \varphi(y)\\ &= (1 - t)\|y - z_n\| + J(z_n) - \varphi(y)\\ &= -t\|y - z_n\| + \big[J(z_n) + \|y - z_n\| - \varphi(y)\big].\end{aligned}$$

Let $t_n = 2^{-n} + [J(z_n) + \|y - z_n\| - \varphi(y)]^{1/2}$. Observe that

$$\lim_{n\to\infty} \left[\frac{\varphi[y + t_n(z_n - y)] - \varphi(y)}{t_n} \langle x^*, z_n - y\rangle\right] = 0.$$

We have that

$$\lim_{n\to\infty}\inf\bigl[-\|y-z_n\|+\langle x^*, y-z_n\rangle+t_n\bigr]\geqslant 0$$

and

$$\begin{aligned}\lim_{n\to\infty}\sup\|y-z_n\| &\leqslant \lim_{n\to\infty}\inf\langle x^*, y-z_n\rangle \leqslant \lim_{n\to\infty}\sup\langle x^*, y-z_n\rangle\\ &\leqslant \lim_{n\to\infty}\inf\|y-z_n\| \leqslant \lim_{n\to\infty}\sup\|y-z_n\|,\end{aligned}$$

which again shows $\|x^*\|\geqslant 1$. Thus $\|x^*\|=1$ and

$$\lim_{n\to\infty}\|y-z_n\|=\lim_{n\to\infty}\langle x^*, y-z_n\rangle. \tag{2}$$

By (1) and (2), we have that $\lim_{n\to\infty}[J(z_n)+\langle x^*, y-z_n\rangle]=\varphi(y)$.

Thus for each $\varepsilon>0$, there is $\delta>0$ such that whenever $z\in Z(y,\delta)$, it follows that

$$J(z)+\langle x^*, y-z\rangle>\left(1-\frac{\varepsilon}{2}\right)\varphi(y).$$

This implies $y\in\Omega(Z)\cap X_0$. Since $\|y-x_0\|<\varepsilon$, this establishes our density assertion. □

Theorem 2.1. *If Z is a closed, boundedly relatively weak compact, nonempty subset of a Banach space X, X_0 is a convex subset in X and X_0 has approximatively Z-property* (K)*, then the set $M(Z)\cap X_0$ is a dense G_δ-subset of $X_0\setminus Z$.*

Proof. From Lemmas 2.3 and 2.5, we have that $L(Z)\cap X_0$ is a dense G_δ-subset of $X_0\setminus Z$. Let $x\in L(Z)\cap X_0$. Then for each n, there exist $\delta_n>0$ and $x_n^*\in X^*$ satisfying $J(z)+\langle x_n^*, x-z\rangle>(1-2^{-n})\varphi(x)$ for all $z\in Z(x,\delta_n)$. Now choose any sequence $\{z_n\}\subset Z$ satisfying $\lim_{n\to\infty}[J(z_n)+\|x-z_n\|]=\varphi(x)$. Without loss of generality, we may suppose $J(z_n)+\|x-z_n\|<\varphi(x)+1/n$ for all $n\in N$. Thus, $0\leqslant J(z_n)<\varphi(x)+1$, the sequence $\{z_n\}\subset Z$ is bounded. Since Z is a boundedly relatively weakly compact subset of a Banach space X, it will contain a sequence weakly convergent to an element $z\in X$. Therefore, without loss of generality, we can suppose

$$\begin{aligned}&J(z_n)+\|x-z_n\|\to\varphi(x); \qquad x-z_n\xrightarrow{w} x-z_0;\\ &J(z_n)\to\alpha; \qquad \|x-z_n\|\to\beta.\end{aligned}$$

Let $m\in N$ be fixed and let $n_0\in N$ be such that $1/n_0<\delta_m$. Then

$$\begin{aligned}&J(z_n)+\|x-z_n\|<\varphi(x)+\frac{1}{n}<\varphi(x)+\delta_m, \quad \text{so that}\\ &J(z_n)+\langle x^*, x-z_n\rangle>(1-2^{-m})\varphi(x)\end{aligned}$$

for all $n\in N$, $n\geqslant n_0$, which yields, for $n\to\infty$,

$$\alpha\bigl\|x_m^*\bigr\|\cdot\|x-z_0\|\geqslant\alpha+\bigl\langle x_m^*, x-z_0\bigr\rangle\geqslant(1-2^{-m})\varphi(x).$$

Since $\|x_m^*\|=1$, we obtain (letting $m\to\infty$)

$$\alpha+\|x-z_0\|\geqslant\varphi(x). \tag{3}$$

By the weakly lower semicontinuity of the norm and $x - z_n \xrightarrow{w} x - z_0$, we have

$$\|x - z_0\| \leqslant \lim_{n\to\infty} \|x - z_n\| = \beta,$$

which, combined with (3), gives

$$\varphi(x) \leqslant \alpha + \|x - z_0\| \leqslant \alpha + \beta = \varphi(x),$$

implying $\|x - z_0\| = \beta = \lim_{n\to\infty} \|x - z_n\|$.

From $x - z_n \xrightarrow{w} x - z_0$, $\|x - z_n\| \to \|x - z_0\|$ and the approximatively Z-property (K), verified by X_0, one obtains $\lim_{n\to\infty} \|z_n - z_0\| = 0$. As the set Z is closed and $\{z_n\} \subset Z$, it follows that $z_0 \in Z$. The lower semicontinuity of the function J implies $J(z_0) \leqslant \lim_{n\to\infty} J(z_n) = \alpha$. But then $\varphi(x) \leqslant J(z_0) + \|x - z_0\| \leqslant \alpha + \beta = \varphi(x)$, showing that $\varphi(x) = J(z_0) + \|x - z_0\|$ and $\lim_{n\to\infty} \|z_n - z_0\| = 0$. The proof is completed. □

Remark. Using Theorem 2.1 method, we may prove that if J is a weakly lower semicontinuous real-valued functional bounded from below and defined on a weakly compact subset of a Banach space X, then the set of all points x in X for which the problem J-inf has a solution is a dense G_δ-subset of X.

The following corollary is the main result in [5], it extends a result of Lau [11] on nearest points (the case $J \equiv 0$).

Corollary 2.1 [5, Theorem 2.1]. *Let Z be a nonempty closed subset of a reflexive Kadec–Banach space X. Then the set $\{x \in X;$ Problem J-inf has a solution $z_0 \in Z\}$ is a dense G_δ-subset of X.*

Proof. It follows from Theorem 2.1 and from the fact that every closed nonempty subset of a reflexive space is a boundedly relatively weakly compact set. □

Theorem 2.2. *Let Z be a nonempty, boundedly relatively weakly compact, closed subset of a Banach space X. If X_0 is a convex subset in X, X_0 has approximatively Z-property* (K), *and X_0 is approximatively Z-strictly convex. Then the set $N(Z) \cap X_0$ is a dense G_δ-subset of $X_0 \setminus Z$.*

Proof. From Theorem 2.1, it suffices to prove that each point of the dense G_δ-subset $L(Z) \cap X_0$ of $X_0 \setminus Z$ has at most one point $z_0 \in Z$ satisfying $J(z_0) + \|x - z_0\| = \varphi(x)$ for $x \in X_0 \setminus Z$ provided that Z is a closed, boundedly relatively weakly compact, nonempty subset of a Banach space X, and X_0 is approximatively Z-strictly convex. Now let $x \in L(Z) \cap X_0$ and $y, z \in Z$ with $J(z) + \|x - z\| = J(y) + \|x - y\| = \varphi(x)$; then the functional x^* guaranteed by the definition of $L(Z)$ has $\|x^*\| = 1$,

$$J(z) + \langle x^*, x - z\rangle = J(y) + \langle x^*, x - y\rangle = \varphi(x)$$

and

$$\begin{aligned}\|x - y\| + \|x - z\| &\geqslant \|(x - y) + (x - z)\| \geqslant \langle x^*, x - y\rangle + \langle x^*, x - z\rangle \\ &= 2\varphi(x) - J(z) - J(y) = \big[\varphi(x) - J(y)\big] + \big[\varphi(x) - J(z)\big] \\ &= \|x - y\| + \|x - z\|.\end{aligned}$$

By approximatively Z-strict convexity of X_0, $y = z$, as required. By Lemmas 2.3 and 2.5, $L(Z) \cap X_0$ is a dense G_δ-subset of $X_0 \setminus Z$. $\square$

Corollary 2.2. *Suppose that X is a reflexive, Kadec and strictly convex Banach space. Let Z be a closed nonempty subset of X. Then the set*

$$\Big\{x \in X \setminus Z;\ \ \textit{problem } J\textit{-inf has a unique solution } z_0 \in Z \ \textit{and every sequence } \{z_n\} \subset Z \textit{ satisfying } \lim_{n\to\infty}\big[J(z_n) + \|x - z_n\|\big] = \varphi(x) \textit{ converges strongly to } z_0\Big\}$$

is a dense G_δ-subset of $X_0 \setminus Z$.

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